Bosonization of Thirring Model in Arbitrary Dimension

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Abstract

We propose to use a novel master Lagrangian for performing the bosonization of the D-dimensional massive Thirring model in $D = d + 1 \geq 2$ dimensions. It is shown that our master Lagrangian is able to relate the previous interpolating Lagrangians each other which have been recently used to show the equivalence of the massive Thirring model in (2+1) dimensions with the Maxwell-Chern-Simons theory. Starting from the phase-space path integral representation of the master Lagrangian, we give an alternative proof for this equivalence up to the next-to-leading order in the expansion of the inverse fermion mass. Moreover, in (3+1)-dimensional case, the bosonized theory is shown to be equivalent to the massive antisymmetric tensor gauge theory. As a byproduct, we reproduce the well-known result on bosonization of the (1+1)-dimensional Thirring model following the same strategy. Finally a possibility of extending our strategy to the non-Abelian case is also discussed.

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1 Introduction

In this paper we investigate the bosonization of the Thirring model [1] as a gauge theory [2, 3, 4] in $D = d + 1$ spacetime dimensions ($D \geq 2$). As is well known, a lot of works have been devoted to the bosonization of the (1+1)-dimensional fermionic model, see [5, 6, 7, 4] and references therein for the Thirring model. However the bosonization is not necessarily restricted to the (1+1) dimensional case. Actually the fermion-boson equivalence was discussed earlier e.g. in [8]. Moreover the bosonization of fermion systems in $D > 2$ dimensions has regained interest by recent works [9, 10, 2, 4]. Particularly the bosonization recipe for abelian systems in $D = 3$ was devised in [9, 10, 2, 4]. We start from one of the reformulations of the Thirring model as a gauge theory which is first proposed by Itoh et al. [2] and subsequently one of the authors (K.-I. K) [3, 4] from a different viewpoint. The basic idea of introducing the gauge degrees of freedom in the bosonization was earlier proposed in more general form in [7, 9], although the works [2, 3, 4] were done independently. In the vanishing coupling limit $G \to 0$, our reformulation of the Thirring model reduces to the result of [7, 9].

In this paper we consider the Thirring model defined by the Lagrangian:

$$\mathcal{L}_{Th} = \bar{\psi}^a i \gamma^\mu \partial_\mu \psi^a - m_a \bar{\psi}^a \psi^a - \frac{G}{2N}(\bar{\psi}^a \gamma_\mu \psi^a)(\bar{\psi}^b \gamma^\mu \psi^b), \qquad (1.1)$$

where ψ^a is a Dirac spinor and the indices a, b are summed over from 1 to N, and $\gamma_\mu (\mu = 0, ..., D - 1)$ are gamma matrices satisfying the Clifford algebra, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{1} = 2\text{diag}(1, -1, ..., -1)$. As usual, by introducing an auxiliary vector field A_μ, the Thirring model is equivalently rewritten as

$$\mathcal{L}_{Th'} = \bar{\psi}^a i \gamma^\mu D_\mu[A]\psi^a - m_a \bar{\psi}^a \psi^a + \frac{1}{2G}A_\mu A^\mu. \qquad (1.2)$$

where

$$D_\mu[A] \equiv \partial_\mu - \frac{i}{\sqrt{N}}A_\mu. \qquad (1.3)$$

The fermionic degrees of freedom can be integrated away from the Lagrangian (1.2). Especially, in the massive fermion case, the fermion determinant leads to a local expression for $\mathcal{L}_G[A]$ of the field A_μ defined by

$$\ln \det \left(\frac{i\gamma^\mu D_\mu[A] + m_a}{i\gamma^\mu \partial_\mu + m_a}\right) = \int d^3 x \mathcal{L}_G[A] + \mathcal{O}\left(\frac{\partial^2}{|m_a|^2}\right). \qquad (1.4)$$

The original Thirring model (1.1) has no gauge symmetry and this is the case even after the auxiliary field is introduced in (1.2). However, if we are allowed to identify the vector field A_μ with the gauge field and, at the same time, able to adopt an appropriate gauge-invariant regularization scheme, the resulting $\mathcal{L}_G[A]$ in (1.4) leads to the gauge invariant expression. For example, in 2+1 dimensions, we have

$$\mathcal{L}_G[A] = \frac{i\theta_{CS}}{4}\epsilon^{\mu\nu\rho}A_\rho F_{\mu\nu} - \frac{1}{24\pi|m|}F_{\mu\nu}F^{\mu\nu}, \qquad (1.5)$$

with

$$\theta_{CS} = \frac{1}{N} \sum_{a=1}^{N} sgn(m_a) \frac{1}{4\pi}, \tag{1.6}$$

where $|m_a| = m$ for all a is assumed [4]. For this scenario to be successful, the Lagrangian

$$\mathcal{L}_G[A] + \frac{1}{2G} A_\mu^2, \tag{1.7}$$

which is self-dual model, for A_μ must be gauge invariant. This is realized by identifying the Thirring model as a gauge-fixed version of some gauge theory [3] by following the Batalin-Fradkin method [11] based on the general formalism of Batalin-Fradkin-Vilkovisky (BFV) [12] for constraint system. Here the requirement of gauge invariance plays the role of selecting a class of gauge-invariant regularizations and of removing some ambiguities related to the regularization [3].

Keeping the above remarks in mind, we briefly review the recent development on bosonization. It has been shown that the (2+1)-dimensional massive Thirring model is equivalent to the Maxwell-Chern-Simons (MCS) theory, up to the leading order [10] and to the next-to-leading order in $1/|m|$ [4]. [1] This fact was first shown by way of the interpolating Lagrangian [10]. However the interpolating Lagrangians adopted by two papers [10, 4] are different from each other. Fradkin and Schaposnik [10] uses the interpolating Lagrangian of the form:

$$\mathcal{L}_{FS}[V, H] = \frac{1}{2G} V^\mu V_\mu - \frac{1}{2} \epsilon^{\mu\nu\rho} V_\rho F_{\mu\nu}[H] + 2\pi \epsilon^{\mu\nu\rho} H_\mu \partial_\nu H_\rho, \tag{1.8}$$

where $F_{\mu\nu}[H]$ is the field strength for the gauge field H_μ. They start from the observation that the Lagrangian of the Thirring model written in terms of the auxiliary field V_μ (corresponding to A_μ in eq. (1.4)) up to the leading order of $1/m$ is equal to the self-dual Lagrangian introduced by Townsend, Pilch and van Nieuwenhuizen [14] and Deser and Jackiw [15]:

$$\mathcal{L}_{SD}[V] = \frac{1}{2G} V_\mu V^\mu + \mathcal{L}_G[V]. \tag{1.9}$$

Then they find the master Lagrangian (1.8) such that the self-dual Lagrangian (1.9) is obtained by integrating out the H_μ field. The MCS theory with the Lagrangian

$$\mathcal{L}_{MCS}[H] = -\frac{G}{4} F_{\mu\nu}[H] F^{\mu\nu}[H] - 2\pi \epsilon^{\mu\nu\rho} H_\mu \partial_\nu H_\rho, \tag{1.10}$$

is obtained by integrating away the field V_μ from $\mathcal{L}_{FS}[V, H]$. It should be noted that the interpolating Lagrangian (1.8) is invariant under the transformation: $\delta H_\mu = \partial_\mu \omega$, $\delta V_\mu = 0$. However the field V_μ does not have the gauge invariance,

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[1]Note that the bosonization of free fermion model in (2+1) dimensions reduces to the Chern-Simons theory (without the Maxwell term) [9]. In our formalism, this case is reproduced as the free fermion limit, $G \to 0$.

although they use the gauge invariant expression (1.4) for $\mathcal{L}_G[V]$ as if the field V_μ was the gauge field.

On the other hand, Kondo's strategy [4] is in sharp contrast to the treatment of Fradkin and Schaposnik [10]. He adopts the interpolating Lagrangian:

$$\mathcal{L}_K[A, H] = -\frac{G}{4} F_{\mu\nu}[H] F^{\mu\nu}[H] + \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu}[H] A_\rho + \mathcal{L}_G[A], \qquad (1.11)$$

which was *derived* by starting from the reformulation of the Thirring model as a gauge theory. As a result, the interpolating Lagrangian $\mathcal{L}_K[A, H]$ is invariant under two independent gauge transformations: $\delta A_\mu = \partial_\mu \lambda$, and $\delta H_\mu = \partial_\mu \omega$. [2] However the connection of two approaches was not necessarily clear at that stage.

In this paper we propose to use the following master Lagrangian in order to investigate the bosonization of the D-dimensional massive Thirring model ($D \geq 2$):

$$\mathcal{L}_M[A, H, K] = \frac{1}{2G} (A_\mu - K_\mu)^2 + \frac{1}{2} \epsilon^{\mu_1 \dots \mu_D} H_{\mu_3 \dots \mu_D} F_{\mu_1 \mu_2}[K] + \mathcal{L}_G[A], \qquad (1.12)$$

where $H_{\mu_3 \dots \mu_D}$ is anti-symmetric tensor field of rank $D - 2$ for $D > 3$, vector field H_μ for $D = 3$ and a scalar field H for $D = 2$. After redefinition of the field variable, the master Lagrangian (1.12) has another form:

$$\mathcal{L}'_M[A, H, V] = \frac{1}{2G} (V_\mu)^2 + \frac{1}{2} \epsilon^{\mu_1 \dots \mu_D} H_{\mu_3 \dots \mu_D} F_{\mu_1 \mu_2}[V + A] + \mathcal{L}_G[A]. \qquad (1.13)$$

An advantage of the master Lagrangian (1.12) or (1.13) is that it is able to interpolate two types of apparently different interpolating Lagrangians: for example, (1.8) and (1.11) in (2+1) dimensions. Actually, we show from the master Lagrangian (1.13) for $D = 3$ that $\mathcal{L}_{FS}[V, H]$ is obtained by integrating out the A_μ field, while $\mathcal{L}_K[A, H]$ is obtained by integrating out the V_μ field. On the other hand, after eliminating the field H_μ and V_μ, we get (see Figure 1):

$$\mathcal{L}_{Th''}[A, \theta] = \frac{1}{2G} (A_\mu - \sqrt{N} \partial_\mu \theta)^2 + \mathcal{L}_G[A]. \qquad (1.14)$$

This is nothing but a gauge-invariant formulation of the Thirring model with the field θ being identified with the Stückelberg field. This Lagrangian was the starting point of the gauge-invariant formulation. The Lagrangian (1.14) should be compared with the self-dual Lagrangian (1.9). This difference comes from the fact that our master Lagrangian (1.13) has independent gauge invariance for two *gauge* fields A_μ and H_μ, while this is not the case for V_μ:

$$\delta A_\mu = \partial_\mu \lambda, \quad \delta H_\mu = \partial_\mu \omega, \quad \delta V_\mu = 0. \qquad (1.15)$$

The classical equivalence of the master Lagrangian with the non-linear σ-model is easy to understand. Indeed the master Lagrangian (1.13) is a polynomial formulation of the *gauged* non-linear σ-model [16, 17, 18]:

$$\mathcal{L}_{gNL\sigma}[\varphi, A] = (D_\mu[A]\varphi)^\dagger (D^\mu[A]\varphi) + \mathcal{L}_G[A], \qquad (1.16)$$

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[2] Quite recently the interpolating Lagrangian (1.11) was used to show the equivalence to all orders in the inverse fermion mass by Banerjee [13].

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with a local constraint: $\varphi(x)\varphi^*(x) = \frac{N}{2G}$. [3]

Nevertheless, it is not necessarily straightforward to show the quantum equivalence. In particular, we must be rather careful in treating the non-Abelian case, which will be discussed in the final section. The bosonization [9, 10, 2, 4] in $D > 2$ dimensions has been carried out based on the configuration space path-integral expressions of the partition functions. In this paper we show the equivalence between the (2+1)-dimensional massive Thirring model and the Maxwell-Chern-Simons theory by starting from the phase-space path integral representation of our master Lagrangian. This equivalence is shown up to the leading order of $1/m$ in section 2.1 and up to the next-to-leading in section 2.2. This type of investigation is very important to elucidate the constraint structure of the various Lagrangian in question. Such a strategy was taken in [23] for the master Lagrangian of Deser and Jackiw [15] to study the connection between the self-dual model and the Maxwell-Chern-Simons theory, which is now included as a part of our investigation in the following. In section 3, our method is applied to the (1+1) dimensional case and we reproduce the previous result on the bosonization of (1+1)-dimensional Thirring model. The case of $D \geq 4$ is discussed in section 4. The final section is devoted to conclusion and discussion.

2　(2+1) dimensions

2.1　up to the leading order

In order to demonstrate the relation among the massive Thirring model, the self-dual model, the Maxwell-Chern-Simons theory and the non-linear σ model, we take into account $\mathcal{L}_G(A)$ defined in (1.5) up to the leading order of $\frac{1}{m}$:

$$\mathcal{L}_G[A] = \frac{i\theta_{CS}}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho. \tag{2.1}$$

Hence the master Lagrangian (1.13) up to the leading order reads

$$\mathcal{L}_{3Leading}[A, H, V] = \frac{i\theta_{CS}}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{2G}V_\mu V^\mu + \frac{1}{2}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho}[A + V]. \tag{2.2}$$

The Lagrangian (2.2) has the primary constraints:

$$\phi_A^0 \equiv \pi_A^0 \approx 0, \quad \phi_A^i \equiv \pi_A^i - \epsilon^{ij}(\frac{i\theta_{CS}}{2}A_j + H_j) \approx 0,$$

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[3] As a special case, putting $A_\mu = 0$ in the master Lagrangian, we obtain the polynomial formulation

$$\mathcal{L}_P[H, K] = \frac{1}{2G}(K_\mu)^2 + \frac{1}{2}\epsilon^{\mu_1\cdots\mu_D}H_{\mu_3\ldots\mu_D}F_{\mu_1\mu_2}[K], \tag{1.17}$$

of the non-linear σ-model [20, 21, 22] with the Lagrangian

$$\mathcal{L}_{NL\sigma}[\varphi] = (\partial_\mu\varphi)^\dagger(\partial^\mu\varphi). \tag{1.18}$$

$$\phi_V^0 \equiv \pi_V^0 \approx 0, \quad \phi_V^i \equiv \pi_V^i - \epsilon^{ij} H_j \approx 0,$$
$$\phi_H^\mu \equiv \pi_H^\mu \approx 0, \tag{2.3}$$

where $\epsilon^{ij} \equiv \epsilon^{0ij}$. Then we obtain the canonical Hamiltonian:

$$\mathcal{H}_{3Leading} = -i\theta_{CS}\epsilon^{ij} A_0 \partial_i A_j - \frac{1}{2G}V^\mu V_\mu$$
$$-\frac{1}{2}\epsilon^{ij} H_0 F_{ij}[A+V] - \epsilon^{ij} H_i \partial_j (A_0 + V_0). \tag{2.4}$$

Using this Hamiltonian, we get the secondary constraints:

$$\Phi_A \equiv \epsilon^{ij}\partial_i(i\theta_{CS}A_j + H_j) \approx 0,$$
$$\Phi_V \equiv \frac{1}{G}V_0 + \epsilon^{ij}\partial_i H_j \approx 0,$$
$$\Phi_H \equiv \frac{1}{2}\epsilon^{ij} F_{ij}[A+V] \approx 0. \tag{2.5}$$

By redefining the constraints:

$$\Phi'_A \equiv \Phi_A + \partial_i \phi_A^i,$$
$$\Phi'_H \equiv \Phi_H + \partial_i \phi_H^i,$$
$$\Phi'_V \equiv \Phi_V + \partial_i \phi_V^i, \tag{2.6}$$

the Poisson brackets are simplified so that the resulting Poisson brackets of Φ'_A, Φ'_H with other constraints vanish:

$$\{\Phi'_A, \cdot\} = \{\Phi'_H, \cdot\} = 0. \tag{2.7}$$

Then we enumerate the non-vanishing Poisson brackets:

$$\{\Phi'_V(\vec{x},t), \phi_V^0(\vec{y},t)\} = \frac{1}{G}\delta^2(\vec{x}-\vec{y}),$$
$$\{\phi_A^i(\vec{x},t), \phi_A^j(\vec{y},t)\} = -i\theta_{CS}\epsilon^{ij}\delta^2(\vec{x}-\vec{y}), \quad \{\phi_A^i(\vec{x},t), \phi_H^j(\vec{y},t)\} = -\epsilon^{ij}\delta^2(\vec{x}-\vec{y}),$$
$$\{\phi_V^i(\vec{x},t), \phi_H^j(\vec{y},t)\} = -\epsilon^{ij}\delta^2(\vec{x}-\vec{y}).$$

Therefore the four constraints $\phi_A^0, \phi_H^0, \Phi'_A, \Phi'_H$ are first class and the eight constraints $\phi_V^0, \Phi_V, \phi_A^i, \phi_V^i, \phi_H^i$ are second class. For the first class constraints $\phi_A^0, \phi_H^0, \Phi'_A, \Phi'_H$, we choose the gauge fixing conditions:

$$\chi_A^0 \equiv A^0 = 0, \quad \chi_A \equiv \frac{1}{2}\partial_i A^i - \frac{1}{i\theta_{CS}}\epsilon_{ij}\partial^i(\pi_A^j - \pi_V^j) = 0,$$
$$\chi_H^0 \equiv H^0 = 0, \quad \chi_H \equiv -\epsilon_{ij}\partial^i\pi_V^j = 0. \tag{2.8}$$

The Poisson brackets among the first class constraints and gauge fixing conditions are evaluated:

$$\{\chi_A^0(\vec{x},t), \phi_A^0(\vec{y},t)\} = \delta^2(\vec{x}-\vec{y}),$$
$$\{\chi_H^0(\vec{x},t), \phi_H^0(\vec{y},t)\} = \delta^2(\vec{x}-\vec{y}),$$
$$\{\chi_A(\vec{x},t), \Phi'_A(\vec{y},t)\} = -\partial_i^2\delta^2(\vec{x}-\vec{y}),$$
$$\{\chi_H(\vec{x},t), \Phi'_H(\vec{y},t)\} = -\partial_i^2\delta^2(\vec{x}-\vec{y}), \tag{2.9}$$

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and the others are zero. Then we get a set of constraints and gauge fixing conditions, all of which are second class.

Now, we get the master partition function in phase space as [19]

$$Z_M^{3Leading} = \int \mathcal{D}A_\mu \mathcal{D}V_\mu \mathcal{D}H_\mu \mathcal{D}\pi_A^\mu \mathcal{D}\pi_V^\mu \mathcal{D}\pi_H^\mu \prod \delta(\phi)\delta(\Phi)\delta(\chi)$$
$$\times \exp\{i \int d^3x (\pi_A^\mu \dot{A}_\mu + \pi_V^\mu \dot{V}_\mu + \pi_H^\mu \dot{H}_\mu - \mathcal{H}_{3Leading})\}, \quad (2.10)$$

where

$$\prod \delta(\phi)\delta(\Phi)\delta(\chi) \equiv \delta(\phi_A^0)\delta(\phi_H^0)\delta(\phi_V^0)\delta(\chi_A^0)\delta(\chi_H^0)\delta(\chi_A)\delta(\chi_H)$$
$$\times \delta(\Phi_A')\delta(\Phi_H')\delta(\Phi_V') \prod_i \delta(\phi_A^i)\delta(\phi_V^i)\delta(\phi_H^i). \quad (2.11)$$

We can easily perform the integration over A^0, H^0 and all the momentum variables $\pi_A^\mu, \pi_V^\mu, \pi_H^\mu$. Thus, we obtain the master partition function in the configuration space up to the leading order:

$$Z_M^{3Leading} = \int \mathcal{D}A_\mu \mathcal{D}V_\mu \mathcal{D}H_\mu \delta(\partial_i A^i)\delta(\partial_i H^i)\delta(\frac{1}{G}V_0 + \epsilon^{ij}\partial_i H_j)$$
$$\times \exp\{i \int d^3x \mathcal{L}_{3Leading}[A, H, V]\}, \quad (2.12)$$

where we have exponentiated the constraints Φ_A', Φ_H' and recovered A_0, H_0 by identifying the Lagrange multiplier fields for Φ_A', Φ_H' with A_0, H_0 respectively. We notice that, in the partition function in configuration space (2.12), the original Lagrangian (2.2) is recovered.

From the partition function (2.12), we can show that Kondo's interpolating Lagrangian $\mathcal{L}_K$ (1.11) appears after V_μ integration, while Fradkin-Schaposnik's interpolating Lagrangian $\mathcal{L}_{FS}$ (1.8) appears after A_μ integration as follows.

At first, we integrate out the V_μ field. The integration can be easily performed and the result is

$$Z_M^{3Leading} = \int \mathcal{D}A_\mu \mathcal{D}H_\mu \delta(\partial_i A^i)\delta(\partial_i H^i) \exp\{i \int d^3x \mathcal{L}_K[A, H]\}, \quad (2.13)$$

$$\mathcal{L}_K[A, H] \equiv -\frac{G}{4}F^{\mu\nu}[H]F_{\mu\nu}[H] + \frac{1}{2}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho}[A]$$
$$+\frac{i\theta_{CS}}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho. \quad (2.14)$$

The Lagrangian $\mathcal{L}_K$ is the same as (1.11) up to the leading order. Using the interpolating Lagrangian (2.14), we can show the equivalence of the Maxwell-Chern-Simons theory and the self-dual model, as carried out in [4].

Next, we want to perform the A_μ integration. The A_0 integration gives

$$Z_M^{3Leading} = \int \mathcal{D}A_i \mathcal{D}V_\mu \mathcal{D}H_\mu \delta(\partial_i A^i)\delta(\partial_i H^i)\delta(\frac{1}{G}V_0 + \epsilon^{ij}\partial_i H_j)$$
$$\times \delta(\epsilon^{ij}\partial_i(i\theta_{CS}A_j + H_j)) \exp\{i \int d^3x \mathcal{L}'\}, \quad (2.15)$$

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$$\mathcal{L}' \equiv \frac{1}{2}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho}[V] + \frac{1}{2G}V^\mu V_\mu + \frac{1}{2}\epsilon^{ij}H_0 F_{ij}[A]$$
$$-\epsilon^{ij}H_i\partial_0 A_j - \frac{i\theta_{CS}}{2}\epsilon^{ij}A_i\partial_0 A_j. \tag{2.16}$$

Here A_i field should obey the two constrains:

$$\partial_i A^i = 0, \quad \epsilon^{ij}\partial_i A_j = \frac{i}{\theta_{CS}G}\epsilon^{ij}\partial_i H_j. \tag{2.17}$$

In order to transform the second inhomogeneous constraint to the homogeneous one, we shift the field as

$$A_i = A_i^{cl} + A_i',$$
$$A_i^{cl}(\vec{x},t) \equiv \epsilon_{ij}\partial^j \int D(\vec{x} - \vec{y})\frac{i}{\theta_{CS}G}\epsilon^{ij}\partial_i H_j(\vec{y},t)d\vec{y},$$
$$\partial_i^2 D(\vec{x}) = \delta^2(\vec{x}). \tag{2.18}$$

As a result, the A_i' field satisfies the homogeneous constraints:

$$\partial^i A_i' = 0, \qquad \epsilon^{ij}\partial_i A_j' = 0, \tag{2.19}$$

and the partition function reads

$$Z_M^{3Leading} = \int \mathcal{D}A_i'\mathcal{D}V_\mu\mathcal{D}H_\mu\delta(\partial^i A_i')\delta(\partial_i H^i)\delta(\epsilon^{ij}\partial_i A_j')$$
$$\times\delta(\frac{1}{G}V_0 + \epsilon^{ij}\partial_i H_j)\exp\{i\int d^3x\mathcal{L}''\}, \tag{2.20}$$
$$\mathcal{L}'' \equiv \frac{1}{2}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho}[V] + \frac{1}{2G}V^\mu V_\mu + \epsilon^{ij}\partial_0(A_i^{cl} + A_i')H_j$$
$$-\frac{1}{i\theta_{CS}}\epsilon^{ij}H_0\partial_i H_j + \frac{i\theta_{CS}}{2}\epsilon^{ij}\partial_0(A_i^{cl} + A_i')(A_j^{cl} + A_j'). \tag{2.21}$$

Moreover we integrate out A_i' and obtain

$$Z_M^{3Leading} = \int \mathcal{D}V_\mu\mathcal{D}H_\mu\delta(\partial_i H^i)\delta(\frac{1}{G}V_0 + \epsilon^{ij}\partial_i H_j)$$
$$\times \exp\{i\int d^3x\mathcal{L}_{FS}[H,V]\}, \tag{2.22}$$
$$\mathcal{L}_{FS}[H,V] \equiv \frac{i}{2\theta_{CS}}\epsilon^{\mu\nu\rho}H_\mu\partial_\nu H_\rho + \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}[V]H_\rho + \frac{1}{2G}V^\mu V_\mu. \tag{2.23}$$

The Lagrangian $\mathcal{L}_{FS}$ coincides with the one (1.8) given in ref. [10].

Finally, we integrate out both H_μ and V_μ fields. We perform H_0 integration to result in the delta function $\delta(\epsilon^{ij}F_{ij}[A+V])$. As in the case of A_i integration of (2.15), we shift the H_i field as

$$H_i = H_i^{cl} + H_i',$$
$$H_i^{cl}(\vec{x},t) \equiv -\epsilon_{ij}\partial^j \int D(\vec{x} - \vec{y})\frac{1}{G}V_0(\vec{y},t)d\vec{y}. \tag{2.24}$$

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Then the partition function is rewritten as

$$Z_M^{3Leading} = \int \mathcal{D}A_\mu \mathcal{D}V_\mu \mathcal{D}H_i' \delta(\partial_i A^i) \delta(\epsilon^{ij} F_{ij}[A+V])$$
$$\times \delta(\partial_i H'^i) \delta(\epsilon^{ij} \partial_i H_j') \exp\{i \int d^3x \mathcal{L}'''\}, \qquad (2.25)$$

$$\mathcal{L}''' \equiv -\epsilon^{ij}(H^{cl} + H')_i \partial_0 (A+V)_j - \frac{1}{G}V_0(A+V)_0$$
$$+\frac{i\theta_{CS}}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho + \frac{1}{2G}V^\mu V_\mu. \qquad (2.26)$$

We can transform the integration measure $\mathcal{D}V_i$ as

$$\mathcal{D}V_i \delta(\epsilon^{ij} F_{ij}[A+V]) = \mathcal{D}\theta, \qquad (2.27)$$

where we have used the solution $V_i = \sqrt{N}\partial_i\theta - A_i$ for the pure gauge constraint $\epsilon^{ij} F_{ij}[A+V] = 0$. After the residual H_i' and V_0 integration, we arrive at

$$Z_M^{3Leading} = \int \mathcal{D}A_\mu \mathcal{D}\theta \delta(\partial_i A^i) \exp\{i \int d^3x \mathcal{L}_{Th''}[A,\theta]\}, \qquad (2.28)$$

where $\mathcal{L}_{Th''}$ is the gauge-invariant Lagrangian (1.14) of the Thirring model in which the Stückelberg field is introduced to recover the gauge invariance.

If we change the variable θ to

$$\varphi \equiv \sqrt{\frac{N}{2G}} e^{i\theta}, \qquad (2.29)$$

then we obtain

$$Z_M^{3Leading} = \int \mathcal{D}A_\mu \mathcal{D}\varphi \delta(\partial_i A^i) \exp\{i \int d^3x \mathcal{L}_{3gNL\sigma}[A,\varphi]\}, \qquad (2.30)$$

$$\mathcal{L}_{3gNL\sigma}[A,\varphi] = (D_\mu[A]\varphi)^\dagger (D^\mu[A]\varphi) + \frac{i\theta_{CS}}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho. \qquad (2.31)$$

This model (2.31) is nothing but the gauged non-linear σ model, if we identify θ as a phase variable of the scalar field φ. We should remark that the phase variable θ of the scalar field φ can be divided into two parts, one of which is the multi-valued function corresponding to the topologically nontrivial sector and another is a single-valued function describing the fluctuation around a given topological sector [24]. In this paper we take into account the single-valued function only. Apart from this subtlety, we have shown the equivalence of the massive Thirring model to the non-linear σ model.

2.2 up to the next-to-leading order

Next, we examine the master Lagrangian up to the next-to-leading order of $\frac{1}{m}$, because the canonical structure is different from that of the leading order. Up to the next-to-leading order, the master Lagrangian is given by

$$\mathcal{L}_{3Next}[A,H,V] = \frac{i\theta_{CS}}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho - \frac{1}{24\pi|m|}F^{\mu\nu}[A]F_{\mu\nu}[A]$$
$$+\frac{1}{2G}V_\mu V^\mu + \frac{1}{2}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho}[A+V]. \qquad (2.32)$$

From the Lagrangian, we get the primary constraints:

$$\phi_A^0 \equiv \pi_A^0 \approx 0,$$
$$\phi_V^0 \equiv \pi_V^0 \approx 0, \quad \phi_V^i \equiv \pi_V^i - \epsilon^{ij} H_j \approx 0,$$
$$\phi_H^\mu \equiv \pi_H^\mu \approx 0, \tag{2.33}$$

and the canonical Hamiltonian:

$$\mathcal{H}_{3Next} = 3\pi |m| \{\pi_A^i - \epsilon_{ij}(\frac{i\theta_{CS}}{2} A^j + H_j)\}^2 + \pi_A^i \partial_i A_0$$
$$+ \frac{1}{24\pi|m|} F^{ij}[A] F_{ij}[A] - \frac{1}{2G} V^\mu V_\mu - \frac{1}{2} H_0 \epsilon^{ij} F_{ij}[A+V]$$
$$- \epsilon^{ij} H_i \partial_j V_0 - \frac{i\theta_{CS}}{2} \epsilon^{ij} A_0 \partial_i A_j - 3\pi i\theta_{CS} |m| H^i A_i, \tag{2.34}$$

From the Hamiltonian, we get the secondary constraints:

$$\Phi_A \equiv \partial_i(\pi_A^i + \frac{i\theta_{CS}}{2} \epsilon^{ij} A_j) \approx 0,$$
$$\Phi_V \equiv \frac{1}{G} V_0 + \epsilon^{ij} \partial_i H_j \approx 0,$$
$$\Phi_H \equiv \frac{1}{2} \epsilon^{ij} F_{ij}[A+V] \approx 0. \tag{2.35}$$

It turns out that the canonical structure up to the next-to leading order is different from that in the leading order, because all π_A^μ's up to the leading order are constrained. Following the same steps as in the leading-order case, we can see that the the four constraints $\phi_A^0, \Phi_A, \phi_H^0, \Phi_H$ are first class and the six constraints $\phi_V^0, \phi_V^i, \Phi_V, \phi_H^i$ are second class. For these first class constraints, we choose the gauge fixing conditions:

$$\chi_A^0 \equiv A^0 = 0, \quad \chi_A \equiv \partial_i A^i = 0,$$
$$\chi_H^0 \equiv H^0 = 0, \quad \chi_H \equiv \epsilon_{ij} \partial^i \pi_V^j = 0. \tag{2.36}$$

It can be easily shown that the first class constraints $\phi_A^0, \Phi_A, \phi_H^0, \Phi_H$ and the gauge fixing conditions $\chi_A^0, \chi_A, \chi_H^0, \chi_H$ form a set of second class constraints. Then, we can obtain the master partition function in the phase space:

$$Z_M^{3Next} = \int \mathcal{D}A_\mu \mathcal{D}V_\mu \mathcal{D}H_\mu \mathcal{D}\pi_A^\mu \mathcal{D}\pi_V^\mu \mathcal{D}\pi_H^\mu \prod \delta(\phi)\delta(\Phi)\delta(\chi)$$
$$\times \exp\{i \int d^3x (\pi_A^\mu \dot{A}_\mu + \pi_V^\mu \dot{V}_\mu + \pi_H^\mu \dot{H}_\mu - \mathcal{H}_{3Next})\}, \tag{2.37}$$

where

$$\prod \delta(\phi)\delta(\Phi)\delta(\chi) \equiv \delta(\phi_A^0)\delta(\phi_H^0)\delta(\phi_V^0)\delta(\chi_A^0)\delta(\chi_H^0)\delta(\chi_A)\delta(\chi_H)$$
$$\times \delta(\Phi_A)\delta(\Phi_H)\delta(\Phi_V) \prod_i \delta(\phi_V^i)\delta(\phi_H^i). \tag{2.38}$$

As in the leading-order case, we can perform the integration over A_0, H_0 and all the momentum variables $\pi_A^\mu, \pi_V^\mu, \pi_H^\mu$, and exponentiate the constrains Φ_A and Φ_H

by identifying the Lagrange multiplier fields for Φ_A, Φ_H with A_0, H_0 respectively. As a result, we obtain the master partition function in configuration space up to the next-to-leading order:

$$Z_M^{3Next} = \int \mathcal{D}A_\mu \mathcal{D}V_\mu \mathcal{D}H_\mu \delta(\partial_i A^i) \delta(\partial_i H^i) \delta(\frac{1}{G}V_0 + \epsilon^{ij}\partial_i H_j)$$
$$\times \exp\{i \int d^3 x \mathcal{L}_{3Next}[A, H, V]\}. \tag{2.39}$$

By using the same method as in the previous section, we can easily get the partition function with the interpolating Lagrangian of [4] after V_μ integration:

$$Z_M^{3Next} = \int \mathcal{D}A_\mu \mathcal{D}H_\mu \delta(\partial_i A^i) \delta(\partial_i H^i) \exp\{i \int d^3 x \tilde{\mathcal{L}}_K[A, H]\} \tag{2.40}$$

$$\tilde{\mathcal{L}}_K[A, H] \equiv \frac{i\theta_{CS}}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho - \frac{1}{24\pi|m|}F^{\mu\nu}[A]F_{\mu\nu}[A]$$
$$+\frac{1}{2}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho}[A] - \frac{G}{4}F^{\mu\nu}[H]F_{\mu\nu}[H]. \tag{2.41}$$

On the other hand, after the A_μ integration, we obtain the partition function:

$$Z_M^{3Next} = \int \mathcal{D}V_\mu \mathcal{D}H \delta(\partial_i H^i) \delta(\frac{1}{G}V^0 + \epsilon^{ij}\partial_i H_j)$$
$$\times \exp\{i \int d^3 x \tilde{\mathcal{L}}_3[H, V]\}, \tag{2.42}$$

$$\tilde{\mathcal{L}}_3[H, V] = +\frac{i}{2\theta_{CS}}\epsilon^{\mu\nu\rho}H_\mu \partial_\nu H_\rho + \frac{1}{24\pi\theta_{CS}^2|m|}F^{\mu\nu}[H]F_{\mu\nu}[H]$$
$$+\frac{1}{2G}V^\mu V_\mu + \frac{1}{2}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho}[V] + \mathcal{O}(\frac{\partial^2}{m^2}). \tag{2.43}$$

This is an extension of the interpolating Lagrangian (1.8) into the next-to-leading order.

Finally, we can integrate out both H_μ and V_μ fields as in the leading-order case and get

$$Z_M^{3Next} = \int \mathcal{D}A_\mu \mathcal{D}\theta \delta(\partial_i A^i) \exp\{i \int d^3 x \mathcal{L}_{Th''}[A, \theta]\}, \tag{2.44}$$

where $\mathcal{L}_{Th''}$ is the gauge-invariant Lagrangian (1.14) of the Thirring model up to the next-to-leading order.

Following the argument given in the previous subsection, the partition function of the Lagrangian (2.32) is identified with that of the gauged non-linear σ model:

$$Z_M^{3Next} = \int \mathcal{D}A_\mu \mathcal{D}\varphi \delta(\partial_i A^i) \exp\{i \int d^3 x \tilde{\mathcal{L}}_{3gNL\sigma}[A, \varphi]\}, \tag{2.45}$$

$$\tilde{\mathcal{L}}_{3gNL\sigma}[A, \varphi] = (D_\mu[A]\varphi)^\dagger (D^\mu[A]\varphi) + \frac{i\theta_{CS}}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho$$
$$-\frac{1}{24\pi|m|}F^{\mu\nu}[A]F_{\mu\nu}[A]. \tag{2.46}$$

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3 (1+1) dimensions

In 1+1-dimensional case, we can show the equivalence between the massive Thirring model and a free scalar theory.

The master Lagrangian in 1+1-dimensional space-time is given as

$$\mathcal{L}_2[A, H, V] = -\frac{1}{4\pi m^2} F_{\mu\nu}[A] F^{\mu\nu}[A] + \frac{1}{2G} V_\mu V^\mu + \frac{1}{2} H \epsilon^{\mu\nu} F_{\mu\nu}[A + V]. \qquad (3.1)$$

We should note that the H field is a scalar field in 1+1 dimensional space-time. From the Lagrangian (3.1), we get the primary constraints:

$$\begin{aligned}
\phi_A^0 &\equiv \pi_A^0 \approx 0, \\
\phi_V^0 &\equiv \pi_V^0 \approx 0, \qquad \phi_V^1 \equiv \pi_V^1 - \epsilon^{01} H \approx 0, \\
\phi_H &\equiv \pi_H \approx 0,
\end{aligned} \qquad (3.2)$$

and Hamiltonian:

$$\mathcal{H}_2 = \frac{\pi m^2}{2} (\pi_A^1 + \epsilon_{01} H)^2 + \pi_A^1 \partial_1 A_0 + \epsilon^{01} H \partial_1 V_0 - \frac{1}{2G} V^\mu V_\mu. \qquad (3.3)$$

From the Hamiltonian, we obtain the secondary constraints:

$$\begin{aligned}
\Phi_A &\equiv \partial_1 \pi_A^1 \approx 0, \\
\Phi_V &\equiv \frac{1}{G} V^0 + \epsilon^{01} \partial_1 H \approx 0.
\end{aligned} \qquad (3.4)$$

Calculating the Poisson brackets among these constraints $\phi_A^0, \phi_V^0, \phi_V^1, \phi_H, \Phi_A, \Phi_V$, we can see the two constraints ϕ_A^0, Φ_A are first class and the four constraints $\phi_H, \phi_V^1, \phi_V^0, \Phi_V$ are second class. For the first class constraints ϕ_A^0, Φ_A, we choose the gauge fixing conditions:

$$\chi_A^0 \equiv A^0 = 0, \qquad \chi_A \equiv \partial_1 A^1 = 0. \qquad (3.5)$$

Using these constraints and gauge fixing conditions, we obtain the master partition function in phase space:

$$\begin{aligned}
Z_M^2 = \int \mathcal{D}A_\mu \mathcal{D}V_\mu \mathcal{D}H \mathcal{D}\pi_A^\mu \mathcal{D}\pi_V^\mu \mathcal{D}\pi_H \prod \delta(\phi) \delta(\Phi) \delta(\chi) \\
\times \exp\{i \int d^2 x (\pi_A^\mu \dot{A}_\mu + \pi_V^\mu \dot{V}_\mu + \pi_H \dot{H} - \mathcal{H}_2)\},
\end{aligned} \qquad (3.6)$$

where

$$\prod \delta(\phi) \delta(\Phi) \delta(\chi) \equiv \delta(\phi_A^0) \delta(\phi_V^0) \delta(\phi_H) \delta(\phi_V^1) \delta(\Phi_A) \delta(\Phi_V) \delta(\chi_A^0) \delta(\chi_A). \qquad (3.7)$$

We can perform π_H, π_V^0, π_V^1 and A_0 integration and get the master partition function in configuration space:

$$Z_M^2 = \int \mathcal{D}A_\mu \mathcal{D}V_\mu \mathcal{D}H \delta(\partial_1 A^1) \delta(\frac{1}{G}V^0 + \epsilon^{01}\partial_1 H)$$
$$\times \exp\{i \int d^2 x \mathcal{L}_2[A, H, V]\}, \tag{3.8}$$

where we have identified the Lagrange multiplier field for Φ_A with A^0.

Integrating out V_μ field, we get

$$Z_M^2 = \int \mathcal{D}A_\mu \mathcal{D}H \delta(\partial_1 A^1) \exp\{i \int d^2 x \mathcal{L}_{2K}[A, H]\}, \tag{3.9}$$

$$\mathcal{L}_{2K}[A, H] = \frac{G}{2}\partial_\mu H \partial^\mu H - \frac{1}{4\pi m^2}F_{\mu\nu}[A]F^{\mu\nu}[A] + \frac{1}{2}H\epsilon^{\mu\nu}F_{\mu\nu}[A]. \tag{3.10}$$

Indeed, this Lagrangian (3.10) is identical to the interpolating Lagrangian introduced in [4] which shows the equivalence of the massive Thirring model in 1+1-dimensional space-time to the scalar theory with the Lagrangian:

$$\mathcal{L}_{\text{scalar}}[H] = \frac{1}{2}(G + \frac{6\pi}{5})\partial_\mu H \partial^\mu H - 3\pi m^2 H^2 + \mathcal{O}(\frac{1}{m^2}). \tag{3.11}$$

If we perform the integration over the H field in the interpolating Lagrangian (3.10), we expect to get the Lagrangian

$$-\frac{1}{4\pi m^2}F_{\mu\nu}[A]F^{\mu\nu}[A] + \frac{1}{2G}A_\mu^2. \tag{3.12}$$

Nevertheless, we cannot get this result because the interpolating partition function (3.9) is not covariant. In fact, we can obtain the result (3.12) corresponding to (1.7) as mentioned in introduction if we take the covariant gauge-fixing condition.

On the other hand, integrating out the V_μ and H fields, we get the (1+1)-dimensional massive Thirring model:

$$Z_M^2 = \int \mathcal{D}A_\mu \mathcal{D}\theta \delta(\partial_1 A^1) \exp\{i \int d^2 x \mathcal{L}_{2Th}[A, \theta]\} \tag{3.13}$$

$$\mathcal{L}_{2Th}[A, \theta] = \frac{1}{2G}(A_\mu - \sqrt{N}\partial_\mu \theta)^2 - \frac{1}{4\pi m^2}F_{\mu\nu}[A]F^{\mu\nu}[A]. \tag{3.14}$$

Furthermore, the Lagrangian $\mathcal{L}_{2Th}$ is rewritten as

$$\mathcal{L}_{2Th}[A, \theta] = \mathcal{L}_{2gNL\sigma}[A, \varphi]$$
$$\equiv (D_\mu[A]\varphi)^\dagger(D^\mu[A]\varphi) - \frac{1}{4\pi m^2}F_{\mu\nu}[A]F^{\mu\nu}[A], \tag{3.15}$$
$$\varphi \equiv \sqrt{\frac{N}{2G}}e^{i\theta},$$

as well as in the (2+1)-dimensional case. This Lagrangian is that of the gauged non-linear σ model.

4 (d+1) dimensions

We want to perform the same procedure for $D = d + 1 \geq 4$ case, but in this case there appears the reducible constraint. The master Lagrangian in D dimensional space-time is given by

$$\mathcal{L}_D[A, H, V] = \mathcal{L}_G[A] + \frac{1}{2G}V^\mu V_\mu + \frac{1}{2}\epsilon^{\mu_1 \cdots \mu_D} H_{\mu_3 \cdots \mu_D} F_{\mu_1 \mu_2}[A + V], \qquad (4.1)$$

where $H_{\mu_3 \cdots \mu_D}$ is a totally anti-symmetric tensor field of rank $D - 2$. Note that

$$\mathcal{L}_G[A] = -\frac{\kappa_D}{4}F^{\mu\nu}[A]F_{\mu\nu}[A], \qquad (4.2)$$

where κ_D is a divergent constant which depends on regularization-scheme and dimensionality D, see e.g. [4]. From the Lagrangian (4.1), we get the constraints

$$\Phi_H^{i_1 \cdots i_{D-3}} \equiv \epsilon^{i_1 \cdots i_{D-1}} F_{i_{D-2} i_{D-1}}[A + V] \approx 0, \qquad (4.3)$$

as secondary constraints. However, the constraints are not independent, because they satisfy the relations

$$F_1^{i_1 \cdots i_{D-4}} \equiv \partial_{i_{D-3}} \Phi_H^{i_1 \cdots i_{D-3}} = 0, \qquad (4.4)$$

where we should note that the relations are identically zero. By the same logic as above, the relations (4.4) are not independent each other. Therefore the theory with the master Lagrangian (4.1) is a reducible theory of $(D - 3)$-th stage, irrespective of the explicit form of $\mathcal{L}_G[A]$. Even in the reducible theory we can repeat the same treatment as in the previous sections. However, there is no guarantee that the covariant theory is obtained as the final result. If we want to get the covariant result without failure, we must resort to other method, for example, BFV method [12].

4.1 (3+1) dimensions

From the above reason, we treat the (3+1) dimensional system based on the BFV method.

The master Lagrangian in $3 + 1$ dimensional space-time is given by

$$\mathcal{L}_4[A, H, V] = \mathcal{L}_G[A] + \frac{1}{2G}V^\mu V_\mu + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} H_{\mu\nu} F_{\rho\sigma}[A + V], \qquad (4.5)$$

$$\mathcal{L}_G[A] = -\frac{\kappa}{4}F^{\mu\nu}[A]F_{\mu\nu}[A], \qquad (4.6)$$

where $H_{\mu\nu}$ is an anti-symmetric tensor field and κ is a regularization-scheme-dependent divergent constant. From the Lagrangian $\mathcal{L}_4$, we get the primary constraints

$$\phi_A^0 \equiv \pi_A^0 \approx 0,$$

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$$\phi_V^0 \equiv \pi_V^0 \approx 0, \qquad \phi_V^i \equiv \pi_V^i - \epsilon^{ijk} H_{jk} \approx 0,$$
$$\phi_H^{0i} \equiv \pi_H^{0i} \approx 0, \qquad \phi_H^{ij} \equiv \pi_H^{ij} \approx 0. \tag{4.7}$$

and the Hamiltonian

$$\mathcal{H} = \int d^3x \{ -\frac{1}{2\kappa}(\pi_A^i - \epsilon^{ijk} H_{jk})(\pi_{Ai} - \epsilon_{ilm} H^{lm}) - A_0 \partial_i \pi_A^i$$
$$+ \frac{\kappa}{4} F_{ij}[A] F^{ij}[A] - \epsilon^{ijk} H_{0i} F_{jk}[A+V] - \epsilon^{ijk} V_0 \partial_i H_{jk} - \frac{1}{2G} V^\mu V_\mu \}. \tag{4.8}$$

For the constraints ϕ_A^0 and ϕ_H^{0i}, we choose the gauge-fixing conditions

$$A_0 = H_{0i} = 0. \tag{4.9}$$

So, we eliminate the variables (π_A^0, A_0) and (π_H^{0i}, H_{0i}) from now on. The non-vanishing Poisson bracket among the residual constraints is

$$\{\phi_H^{ij}(\vec{x}, t), \phi_V^k(\vec{y}, t)\} = \epsilon^{ijk} \delta^3(\vec{x} - \vec{y}). \tag{4.10}$$

Therefore, the constraints ϕ_V^i and ϕ_H^{ij} are second class. Moreover we shall eliminate these two second class constraints by solving them, and the Poisson bracket changes to the modified one:

$$\{H_{ij}, V_k\} = \frac{1}{2} \epsilon_{ijk}, \tag{4.11}$$

i.e., we can consider that $\epsilon^{ijk} H_{jk}$ and V_i form a canonical pair. From the Hamiltonian (4.8) we get the secondary constraints

$$\Phi_A \equiv \partial_i \pi_A^i \approx 0,$$
$$\Phi_V \equiv \frac{1}{G} V^0 + \epsilon^{ijk} \partial_i H_{jk} \approx 0,$$
$$\Phi_H^i \equiv \epsilon^{ijk} F_{jk}[A+V] \approx 0. \tag{4.12}$$

The non-vanishing Poisson bracket among the residual constraints ϕ_V^0, Φ_A, Φ_V, Φ_H^i is

$$\{\Phi_V(\vec{x}, t), \phi_V^0(\vec{y}, t)\} = \frac{1}{G} \delta^3(\vec{x} - \vec{y}). \tag{4.13}$$

Therefore, the constraints ϕ_V^0, Φ_V are second class constraints and Φ_A, Φ_H^i are first class constraints. In order to treat second class constraints ϕ_V^0, Φ_V, we shall use the Dirac bracket

$$\{A, B\}_D \equiv \{A, B\} - \{A, \phi_V^0\}\{\phi_V^0, \Phi_V\}^{-1}\{\Phi_V, B\} - \{A, \Phi_V\}\{\Phi_V, \phi_V^0\}^{-1}\{\phi_V^0, B\}, \tag{4.14}$$

where A, B are arbitrary variables. Then, it turns out that (π_V^0, V_0) is not a canonical pair and not dynamical, because the Dirac bracket between them is zero:

$$\{\pi_V^0, V_0\}_D = 0. \tag{4.15}$$

From this fact, we concentrate on the first class constraints Φ_A and Φ_H^i in the phase space (π_A^i, A_i), $(\epsilon^{ijk} H_{jk}, V_i)$. However, the first class constraints are not independent, because the constraints Φ_H^i's satisfy the relation

$$\partial_i \Phi_H^i = 0, \tag{4.16}$$

where we note that the relation, i.e., Bianchi identity, is identically zero. In order to get the Lorentz-covariant and locally well-defined quantized action, we should quantize the system without solving the relation (4.16). This can be done when we quantize the system according to the BFV method. So we deal with the system in BFV formalism.

In the BFV formalism, we have to prepare the phase space which consists of the original phase space

$$(\pi_A^i, A_i), \quad (\epsilon^{ijk} H_{jk}, V_i), \tag{4.17}$$

and the extended phase space

	(B, N),		$(\overline{P}, C)$,		$(\overline{C}, P)$,		for $\Phi_A \approx 0$,
Grassmann parity	0	0	1	1	1	1	
ghost number	0	0	-1	1	-1	1	
	(B^i, N_i),		$(\overline{P}^i, C_i)$,		$(\overline{C}^i, P_i)$,		for $\Phi_H^i \approx 0$.
Grassmann parity	0	0	1	1	1	1	
ghost number	0	0	-1	1	-1	1	

$$\tag{4.18}$$

However, the constraints Φ_H^i's satisfy the relation $\partial_i \Phi_H^i = 0$. In this case, we must impose the fermionic constraint

$$\partial_i \overline{P}^i \approx 0, \tag{4.19}$$

and accordingly introduce the canonical pairs

	$(B_{(1)}, N_{(1)})$,		$(\overline{P}_{(1)}, C_{(1)})$,		$(\overline{C}_{(1)}, P_{(1)})$.	
Grassmann parity	1	1	0	0	0	0
ghost number	-1	1	-2	2	-2	2

$$\tag{4.20}$$

This is how to treat the reducible constraints in BFV formalism. To make the Lorentz covariance manifest and perform gauge-fixing of the field $\overline{C}_\mu$, we also introduce the canonical variables

	(B_1^1, N_1^1),		$(\overline{C}_1^1, P_1^1)$,	
Grassmann parity	0	0	1	1
ghost number	0	0	-1	1

$$\tag{4.21}$$

as extra ghost fields. Now, we can construct the BRST charge

$$Q_{BRST} = \int d^3 x \{ C\Phi_A + C_i \Phi_H^i + iC_{(1)} \partial_i \overline{P}^i + PB + P_i B^i + P_{(1)} B_{(1)} + P_1^1 B_1^1 \}. \tag{4.22}$$

From the BRST charge, we can calculate the BRST transformation:

$$
\begin{aligned}
&\delta A_i = \partial_i C, && \delta \pi_A^i = 2\epsilon^{ijk}\partial_j C_k, \\
&\delta H_{ij} = \partial_i C_j - \partial_j C_i, && \\
&\delta V_i = 0, && \delta V_0 = \delta(-G\epsilon^{ijk}\partial_i H_{jk}) = 0,
\end{aligned}
\tag{4.23}
$$

for the variables $A_i, \pi_A^i, H_{ij}, V_\mu$ and

$$
\begin{aligned}
&\delta B = 0, && \delta B^i = 0, && \delta B_{(1)} = 0, && \delta B_1^1 = 0, \\
&\delta N = -P, && \delta N_i = -P_i, && \delta N_{(1)} = P_{(1)}, && \delta N_1^1 = -P_1^1, \\
&\delta \overline{P} = \Phi_A, && \delta \overline{P}^i = \Phi_H^i, && \delta \overline{P}_{(1)} = i\partial_i \overline{P}^i, && \\
&\delta C = 0, && \delta C_i = -i\partial_i C_{(1)}, && \delta C_{(1)} = 0, && \\
&\delta \overline{C} = B, && \delta \overline{C}^i = B^i, && \delta \overline{C}_{(1)} = B_{(1)}, && \delta \overline{C}_1^1 = B_1^1, \\
&\delta P = 0, && \delta P_i = 0, && \delta P_{(1)} = 0, && \delta P_1^1 = 0,
\end{aligned}
\tag{4.24}
$$

for the residual variables (4.18),(4.20) and (4.21). Furthermore, we take the gauge-fixing fermion as

$$
\Psi = \int d^3x\{\overline{C}\chi + \overline{P}N + \overline{C}^i\chi_i + \overline{P}^i N_i + \overline{C}_{(1)}\chi_{(1)} + \overline{P}_{(1)}N_{(1)} + \overline{C}_1^1\chi_1^1\}, \tag{4.25}
$$

where $\chi, \chi_i, \chi_{(1)}, \chi_1^1$ are gauge-fixing functions chosen later. Then, we can give the BRST invariant quantum action

$$
\begin{aligned}
S_q = \int d^4x[&\pi_A^i \dot{A}_i + \epsilon^{ijk}H_{jk}\dot{V}_i + B\dot{N} + B^i\dot{N}_i + B_{(1)}\dot{N}_{(1)} + B_1^1\dot{N}_1^1 + \overline{P}\dot{C} + \overline{P}^i\dot{C}_i \\
&+ \overline{P}_{(1)}\dot{C}_{(1)} + \overline{C}\dot{P} + \overline{C}^i\dot{P}_i + \overline{C}_{(1)}\dot{P}_{(1)} + \overline{C}_1^1\dot{P}_1^1 - \mathcal{H}_c - \{\Psi, Q\}_D],
\end{aligned}
\tag{4.26}
$$

where $\mathcal{H}_c$ is the canonical Hamiltonian

$$
\begin{aligned}
\mathcal{H}_c = &-\frac{1}{2\kappa}(\pi_A^i - \epsilon^{ijk}H_{jk})(\pi_{Ai} - \epsilon_{ilm}H^{lm}) + \frac{\kappa}{4}F_{ij}[A]F^{ij}[A] \\
&-\epsilon^{ijk}V_0\partial_i H_{jk} - \frac{1}{2G}V^\mu V_\mu.
\end{aligned}
\tag{4.27}
$$

Then we can get the master partition function in extended phase space

$$
\begin{aligned}
Z = &\int \mathcal{D}\mu \exp\{iS_q\}, \\
\mathcal{D}\mu \equiv &\mathcal{D}A_i\mathcal{D}H_{ij}\mathcal{D}V_\mu\mathcal{D}\pi_A^i\mathcal{D}B\mathcal{D}N\mathcal{D}\overline{P}\mathcal{D}C\mathcal{D}\overline{C}\mathcal{D}P \\
&\times\mathcal{D}B^i\mathcal{D}N_i\mathcal{D}\overline{P}^i\mathcal{D}C_i\mathcal{D}\overline{C}^i\mathcal{D}P_i\mathcal{D}B_{(1)}\mathcal{D}N_{(1)}\mathcal{D}\overline{P}_{(1)}\mathcal{D}C_{(1)}\mathcal{D}\overline{C}_{(1)}\mathcal{D}P_{(1)} \\
&\times\mathcal{D}B_1^1\mathcal{D}N_1^1\mathcal{D}\overline{C}_1^1\mathcal{D}P_1^1.
\end{aligned}
\tag{4.28}
$$

After we integrate the fields $\pi_A^i, \overline{P}, P, \overline{P}^i, P_i, \overline{P}_{(1)}, P_{(1)}$ and choose the gauge fixing functions $\chi, \chi_i, \chi_{(1)}, \chi_1^1$ as

$$
\chi = \partial^i A_i - \frac{\alpha}{2}B,
$$

$$
\chi_i = \partial^j H_{ji} - \partial_i N_1^1 - \frac{\beta}{2}B_i,
$$

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$$\chi_{(1)} = -i\partial^i C_i - i\gamma P_1^1,$$
$$\chi_1^1 = \partial^i N_i - \frac{\beta}{2} B_1^1, \tag{4.29}$$

the partition function in configuration space is expressed as

$$Z = \int \mathcal{D}A_\mu \mathcal{D}H_{\mu\nu} \mathcal{D}V_\mu \mathcal{D}B \mathcal{D}C \mathcal{D}\overline{C} \mathcal{D}B^\mu \mathcal{D}C_\mu \mathcal{D}\overline{C}^\mu \mathcal{D}B_{(1)} \mathcal{D}N_1^1 \mathcal{D}P_1^1 \mathcal{D}\overline{C}_{(1)} \mathcal{D}C_{(1)}$$
$$\times \exp\{i \int d^4x \mathcal{L}_q'\}, \tag{4.30}$$

$$\mathcal{L}_q' \equiv \mathcal{L}_4[A, H, V] - B(\partial^\mu A_\mu - \frac{\alpha}{2}B) + \overline{C}\partial^\mu \partial_\mu C$$
$$- B^\mu(\partial^\nu H_{\nu\mu} - \partial_\mu N_1^1 - \frac{\beta}{2} B_\mu) + \overline{C}^\mu \partial^\nu(\partial_\nu C_\mu - \partial_\mu C_\nu + \partial_\mu P_1^1)$$
$$+ iB_{(1)}(\partial^\mu C_\mu - \gamma P_1^1) + \overline{C}_{(1)}\partial^\mu \partial_\mu C_{(1)}, \tag{4.31}$$

where α, β, γ are gauge-fixing parameters and we have identified the fields N, N_i, $N_{(1)}$, B_1^1, $\overline{C}_1^1$ with $-A_0, -H_{0i}, -iC_0, B^0, \overline{C}^0$ respectively to recover the Lorentz covariance. [4] After all, the partition function results in

$$Z = \int \mathcal{D}A_\mu \mathcal{D}H_{\mu\nu} \mathcal{D}V_\mu \mathcal{D}B \mathcal{D}B^\mu \mathcal{D}N_1^1 \ \exp\{i\tilde{S}_q + i\tilde{S}_{GF}\}, \tag{4.34}$$

$$\tilde{S}_q[A, H, V] \equiv \int d^4x \mathcal{L}_4[A, H, V], \tag{4.35}$$

$$\tilde{S}_{GF} \equiv \int d^4x \{-B(\partial^\mu A_\mu - \frac{\alpha}{2}B) - B^\nu(\partial^\mu H_{\mu\nu} - \partial_\nu N_1^1 - \frac{\beta}{2}B_\nu)\}, \tag{4.36}$$

where we have integrated out the fields $B_{(1)}, C, \overline{C}, C_\mu, \overline{C}^\mu, P_1^1$. Besides, we should note that the field N_1^1 is still necessary to fix the gauge degrees of freedom for B_μ. Using the master partition function (4.34) in configuration space, we show the equivalence among the various theories as follows.

4.2 $\mathcal{L}[A, H]$, $\mathcal{L}[A]$

At first, we integrate out the V_μ field in (4.34). Then the resulting partition function is

$$Z = \int \mathcal{D}A_\mu \mathcal{D}H_{\mu\nu} \mathcal{D}B \mathcal{D}B_\mu \mathcal{D}N_1^1 \exp\{i \int d^4x \mathcal{L}_4[A, H] + i\tilde{S}_{GF}\}, \tag{4.37}$$

---

[4] Total Lagrangian $\mathcal{L}_q'$ can be written in the BRST invariant form

$$\mathcal{L}_q' = \mathcal{L}_4 + \delta F,$$
$$F \equiv -\overline{C}(\partial^\mu A_\mu - \frac{\alpha}{2}B) - \overline{C}^\mu(\partial^\nu H_{\nu\mu} - \partial_\mu N_1^1 - \frac{\beta}{2}B_\mu) + i\overline{C}_{(1)}(\partial^\mu C_\mu - \gamma P_1^1), \tag{4.32}$$

according to [25]. Here, the covariant BRST transformation δ is defined by

$$\begin{array}{llll}
\delta A_\mu = \partial_\mu C, & \delta H_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu, & \delta V_\mu = 0, & \\
\delta B = 0, & \delta B^\mu = 0, & \delta B_{(1)} = 0, & \delta B_1^1 = 0, \\
\delta C = 0, & \delta C_\mu = -i\partial_\mu C_{(1)}, & \delta C_{(1)} = 0, & \\
\delta \overline{C} = B, & \delta \overline{C}^\mu = B^\mu, & \delta \overline{C}_{(1)} = B_{(1)}, & \delta \overline{C}_1^1 = B_1^1, \\
\delta P_1^1 = 0, & \delta N_1^1 = -P_1^1. & &
\end{array} \tag{4.33}$$

$$\mathcal{L}_4[A,H] \equiv -\frac{\kappa}{4}F^{\mu\nu}[A]F_{\mu\nu}[A] + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}H_{\mu\nu}F_{\rho\sigma}[A] + \frac{G}{12}\tilde{H}^{\mu\nu\rho}\tilde{H}_{\mu\nu\rho}, \quad (4.38)$$

$$\tilde{H}^{\mu\nu\rho} \equiv \partial^\mu H^{\nu\rho} + \partial^\nu H^{\rho\mu} + \partial^\rho H^{\mu\nu} - \partial^\nu H^{\mu\rho} - \partial^\mu H^{\rho\nu} - \partial^\rho H^{\nu\mu}. \quad (4.39)$$

This is an interpolating gauge theory of 3+1 dimensions corresponding to (2.14) or (2.41) of 2+1 dimensions.

Furthermore, if B_μ, N_1^1 and $H_{\mu\nu}$ fields are integrated out, we get the partition function

$$Z = \int \mathcal{D}A_\mu \mathcal{D}B \exp\{i \int d^4x \mathcal{L}_4[A,B]\}, \quad (4.40)$$

$$\mathcal{L}_4[A,B] \equiv -\frac{\kappa}{4}F^{\mu\nu}[A]F_{\mu\nu}[A] + \frac{1}{4G}F^{\mu\nu}[A]\frac{1}{\partial^\rho\partial_\rho}F_{\mu\nu}[A]$$
$$-B(\partial^\mu A_\mu - \frac{\alpha}{2}B). \quad (4.41)$$

This is a gauge theory for the A_μ field with non-local term. Nevertheless, if we choose the gauge-fixing parameter $\alpha = 0$, then the non-local term disappears after the integration of the B field and the partition function changes to

$$Z = \int \mathcal{D}A_\mu \delta(\partial^\mu A_\mu) \exp\{i \int d^4x \mathcal{L}_4[A]\}, \quad (4.42)$$

$$\mathcal{L}_4[A] \equiv -\frac{\kappa}{4}F^{\mu\nu}[A]F_{\mu\nu}[A] - \frac{1}{2G}A_\mu A^\mu. \quad (4.43)$$

This denotes a massive vector theory corresponding to the self-dual model (1.9) in (2+1) dimensions.

4.3 $\mathcal{L}[A,\varphi]$

On the other hand, if we take $\beta = 0$ in (4.34), the B_μ, N_1^1 and $H_{\mu\nu}$ integrations give

$$Z = \int \mathcal{D}A_\mu \mathcal{D}V_\mu \mathcal{D}B \delta(F_{\mu\nu}[A+V]) \exp\{i \int d^4x \mathcal{L}_4[A,V,B]\}, \quad (4.44)$$

$$\mathcal{L}_4[A,V,B] \equiv -\frac{\kappa}{4}F_{\mu\nu}[A]F^{\mu\nu}[A] + \frac{1}{2G}V^\mu V_\mu - B(\partial^\mu A_\mu - \frac{\alpha}{2}B). \quad (4.45)$$

Moreover, we perform the V_μ integration, and obtain the partition function

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\varphi \mathcal{D}B \exp\{i \int d^4x L[A,\varphi,B]\}, \quad (4.46)$$

$$\mathcal{L}[A,\varphi,B] \equiv -\frac{\kappa}{4}F^{\mu\nu}[A]F_{\mu\nu}[A] + \frac{1}{2G}\{(\partial^\mu - iA^\mu)\varphi\}^\dagger\{(\partial^\mu - iA^\mu)\varphi\}$$
$$-B(\partial^\mu A_\mu - \frac{\alpha}{2}B), \quad (4.47)$$
$$\varphi\varphi^* = 1.$$

This is a gauged non-linear σ model.

4.4 $\mathcal{L}[H, V]$

Next, the integration of the A_μ field in (4.34) gives the partition function

$$Z = \int \mathcal{D}H_{\mu\nu}\mathcal{D}V_\mu\mathcal{D}B_\mu\mathcal{D}N_1^1 \exp\{i\int d^4x \mathcal{L}[H, V, B_\mu, N_1^1]\}, \tag{4.48}$$

$$\mathcal{L}[H, V, B_\mu, N_1^1] \equiv \frac{1}{12\kappa}\tilde{H}^{\mu\nu\rho}\frac{1}{\partial^\sigma\partial_\sigma}\tilde{H}_{\mu\nu\rho} + \frac{1}{2G}V^\mu V_\mu + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}H_{\mu\nu}F_{\rho\sigma}[V]$$
$$- B^\nu(\partial^\mu H_{\mu\nu} - \partial_\nu N_1^1 - \frac{\beta}{2}B_\nu). \tag{4.49}$$

This result is α-independent. If we take the gauge-fixing parameter $\beta = 0$, we get

$$Z = \int \mathcal{D}H_{\mu\nu}\mathcal{D}V_\mu\delta(\partial^\mu H_{\mu\nu})\exp\{i\int d^4x \mathcal{L}[H, V]\}, \tag{4.50}$$

$$\mathcal{L}[H, V] \equiv -\frac{1}{\kappa}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2G}V^\mu V_\mu + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}H_{\mu\nu}F_{\rho\sigma}[V], \tag{4.51}$$

after B_μ and N_1^1 integrations. It should be remarked that the non-local term for $H_{\mu\nu}$ changes into a mass term only if we choose $\beta = 0$.

4.5 $\mathcal{L}[H]$

Finally, integration of V_μ in (4.49) leads to

$$Z = \int \mathcal{D}H_{\mu\nu}\mathcal{D}B_\mu\mathcal{D}N_1^1\delta(\partial^\mu H_{\mu\nu})\exp\{i\int d^4x \mathcal{L}[H, B_\mu, N_1^1]\}, \tag{4.52}$$

$$\mathcal{L}[H, B_\mu, N_1^1] \equiv \frac{G}{12}\tilde{H}^{\mu\nu\rho}\tilde{H}_{\mu\nu\rho} + \frac{1}{12\kappa}\tilde{H}^{\mu\nu\rho}\frac{1}{\partial^\sigma\partial_\sigma}\tilde{H}_{\mu\nu\rho}$$
$$- B^\nu(\partial^\mu H_{\mu\nu} - \partial_\nu N_1^1 - \frac{\beta}{2}B_\nu). \tag{4.53}$$

If we take $\beta = 0$, we get

$$Z = \int \mathcal{D}H_{\mu\nu}\delta(\partial^\mu H_{\mu\nu})\exp\{i\int d^4x \mathcal{L}[H]\}, \tag{4.54}$$

$$\mathcal{L}[H] \equiv \frac{G}{12}\tilde{H}^{\mu\nu\rho}\tilde{H}_{\mu\nu\rho} - \frac{1}{\kappa}H^{\mu\nu}H_{\mu\nu}, \tag{4.55}$$

by the integration of B_μ field. As mensioned above, the non-local term changes to a mass term in the case $\beta = 0$. After all, we have bosonized the massive Thirring model to get a tensor gauge theory. This result is consistent with [26]. The result (4.55) can be also obtained from the partition function (4.37) by performing the integrations over all the fields except $H_{\mu\nu}$.

In the general D-dimensional case $(D \geq 4)$, it is expected that the bosonized theory is given by

$$Z = \int \mathcal{D}H_{\mu_1...\mu_{D-2}}\delta(\partial^{\mu_1}H_{\mu1...\mu_{D-2}})\exp\{i\int d^4x \mathcal{L}_D[H]\}, \tag{4.56}$$

$$\mathcal{L}_D[H] \equiv \frac{G}{12}\tilde{H}^{\mu_1...\mu_{D-1}}\tilde{H}_{\mu_1...\mu_{D-1}} - \frac{1}{\kappa_D}H^{\mu_1...\mu_{D-2}}H_{\mu_1...\mu_{D-2}}, \tag{4.57}$$

where $\tilde{H}^{\mu_1\cdots\mu_{D-1}}$ is totally anti-symmetrized one of $\partial^{\mu_1} H^{\mu_2\cdots\mu_{D-1}}$, i.e.,

$$\tilde{H}^{\mu_1\cdots\mu_{D-1}} = (D-2)! \; (\partial^{\mu_1} H^{\mu_2\mu_3\cdots\mu_{D-1}} - \partial^{\mu_2} H^{\mu_1\mu_3\cdots\mu_{D-1}} + \cdots$$
$$+ (-1)^{(D-2)} \partial^{\mu_{D-1}} H^{\mu_1\cdots\mu_{D-2}}). \quad (4.58)$$

5 Conclusion and Discussion

In this paper we have proposed a master Lagrangian (1.12) or (1.13) for performing the bosonization of the Thirring model in arbitrary dimension. Especially, in (2+1) dimensions, this master Lagrangian is able to interpolate the previous two interpolating Lagrangians [10, 4]. Starting from the phase-space path integral formulation of the gauge theory defined by the master Lagrangian, we have shown the equivalence of the (2+1)-dimensional massive Thirring model with the Maxwell-Chern-Simons theory, up to the next-to-leading order of $1/m$. Incidentally it is not difficult to show the equivalence by applying the generalized canonical formalism of Batalin, Fradkin, Vilkovisky and Tyutin [11] to our master Lagrangian, as carried out in the recent work [27] for the self-dual model. Actually, in (3+1)-dimensional case, we have shown based on the BFV method that the bosonized theory of the massive Thirring model is equivalent to the massive antisymmetric tensor theory.

Athough the Thirring model in $D > 2$ dimensions is perturbatively nonrenormalizable, the bosonization technique may throw light on the nonperturbative renormalizability of the Thirring modelin (3+1)-dimensions as the normalizability of the four-fermion interaction in $1/N_f$ expansion. [28] [29]

The most interesting question will be how to generalize our strategy of bosonization into the non-Abelian case. First we remark that it is easy to show the classical equivalence of the non-Abelian-gauged non-linear σ-model with the gauge-invariant formulation of the non-Abelian Thirring model with the Stückelberg field [30]. Indeed the non-Abelian version of our master Lagrangian is at least classically equivalent to the non-Abelian-gauged non-linear σ-model [16, 17, 18]. In this sense our master Lagrangian is easily extended to the non-Abelian case. However, we must specify the gauge-fixing procedure in the master Lagrangian and take into account the ghost field to preserve the BRS symmetry even after the gauge fixing, if we follow the line of [4]. Therefore, in order to show the quantum equivalence of the Thirring model with some kind of gauge theory, we are required to find a clever gauge-fixing so that the redundant fields can be integrated out to arrive at the final gauge theory. Quite recently, nevertheless, it was announced by Bralic et al. [31] that the non-Abelian version of the (2+1)-dimensional Thirring model can be bosonized by following the same strategy as that of Fradkin and Schaposnik [10] with help of the interpolating Lagrangian of the form found in Karlhede et al. [32]. The equivalent gauge theory obtained in [31] is somewhat similar to the Yang-Mills-Chern-Simons theory, but does not exactly coincides with it. However, the questions raised above are not answered in that paper, nor taken up are such questions. In this sense, the bosonization of the

(2+1)-dimensional Thirring model is not yet well understood from our viewpoint of gauge-invariant formulation.

Finally we remark that, in the weak four-fermion coupling limit $G \to 0$, the field V_μ decouples from the master Lagrangian (1.13) after rescaling the field V_μ. In this limit the master Lagrangian reduces to

$$\mathcal{L}'_M[A_\mu, H_\mu] = \frac{1}{2}\epsilon^{\mu_1 \cdots \mu_D} H_{\mu_3 \ldots \mu_D} F_{\mu_1 \mu_2}[A] + \mathcal{L}_G[A_\mu]. \tag{5.1}$$

This should correspond to the free fermion model. Actually this coincides with the result of [7]. If we integrate out the A_μ field, we could perform the bosonization of the free (!) fermion model and would obtain the bosonized theory written in terms of the field H_μ. However this simplified master Lagrangian has the same problems as mentioned above in the presence of the four-fermion interaction. Therefore the essential difficulty of non-Abelian bosonization comes not only from the specific interaction of the original fermionic model but also the gauge-invariance of the master Lagrangian or the hidden gauge-invariance of the free fermionic model. The bosonization of the free fermionic model was performed in the (1+1) dimensional case by Burgess and Quevedo [33]. However, it is not straightforward to extend this analysis to the case of $D > 2$, since they use various peculiarities of (1+1) dimensions. In view of this, a detailed investigation of the non-Abelian versions of the Thirring model will be reported in a subsequent paper [34].

Acknowledgments

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Figure Captions

Fig.1: Equivalence of the various models.

```
                    ┌─────────────────────────────────────────┐
                    │  Master Lagrangian  𝓛[A, H, V]   (1.13)  │
                    └─────────────────────────────────────────┘
```

∫ 𝒟V ∫ 𝒟H

∫ 𝒟V   ∫ 𝒟A

```
┌──────────────────────────────────┐    ┌──────────────────────────────────┐
│          D = 2 (3.10)            │    │       D = 3 (1.8) (2.43)         │
│ 𝓛[A,H]  D = 3 (2.14) (2.41)      │    │ 𝓛[H,V]   D = 4 (4.51)            │
│          D = 4 (4.38)            │    │                                  │
└──────────────────────────────────┘    └──────────────────────────────────┘
```

∫ 𝒟H   ∫ 𝒟A   ∫ 𝒟V

```
                    ┌─────────────────────────────────────┐
                    │  Bosonized Theory   𝓛[H]           │
                    │        D = 2 (3.11)                │
                    │        D = 3 (1.10)                │
                    │        D = 4 (4.55)                │
                    └─────────────────────────────────────┘
```

```
┌──────────────────────────────┐    ┌──────────────────────────────┐
│  Gauged Non-Linear          │    │         D = 2 (3.12)        │
│  σ Model (1.16)             │    │ 𝓛[A]   D = 3 (1.7)          │
│  𝓛[A, φ]                    │    │         D = 4 (4.43)        │
└──────────────────────────────┘    └──────────────────────────────┘
```

∫ 𝒟ψ𝒟ψ̄ │ Auxiliary Field A_μ ∫ 𝒟ψ𝒟ψ̄
          Stückelberg Field θ

                                     Auxiliary Field A_μ

```
              ┌─────────────────────────────────────────┐
              │       Massive Thirring Model           │
              │       𝓛[ψ, ψ̄]   (1.1)                 │
              └─────────────────────────────────────────┘
```